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ON UNSYMMETRICAL ADJUSTMENTS, AND THEIR LIMITS.

BY E. L. DE FOREST.

(*Continued from page* 170, *Vol. VI.*)

THE origin is always at the centre of parallel forces, and this is the position of the vertex, or the maximum ordinate, if the curve is a symmetrical one. The centre of forces for the resultant of k applications of a given formula keeps the same position relatively to the two ends of the resultant formula, as the centre for the given formula had with reference to its two ends. To prove this, let us regard the coefficients of the given formula simply as coefficients of powers of z in the polynomial

$$a_0+a_1z+a_2z^2+\ldots\ldots+a_nz^n, \tag{42}$$

with the condition

$$a_0+a_1+\ldots\ldots+a_n=1. \tag{43}$$

These parallel forces a being supposed to act at right angles to the axis of X and at intervals equal to Δx, let h_1 denote the interval between a_0 and the centre of forces, in other words, the lever arm of the system about the position of a_0 as a fulcrum. Then we have as the condition of equilibrium at the centre of forces,

$$b_1\Delta x = a_0(-h_1)+a_1(-h_1+\Delta x)+\ldots\ldots+a_n(-h_1+n\Delta x)=0,$$

and from this, with the condition (43), we get

$$h_1 = (a_1+2a_2+\ldots\ldots+na_n)\Delta x, \tag{44}$$

which is also evidently the statical moment of the system about the place of a_0. For two applications of the given formula, the resultant coefficients will be those of the powers of z in the square of (42). To form the square we first multiply (42) by a_0, and in this product, taking account of the condition (43), the moment of the coefficients about the place of the first one is a_0h_1. Multiplying next by a_1z we get a product with the moment

$a_1(h_1+\Delta x)$. Likewise multiplying by $a_2 z^2$, the moment for the product is $a_2(h_1+2\Delta x)$, and so on, the total being

$$a_0 h_1+a_1(h_1+\Delta x)+\ldots+a_n(h_1+n\Delta x),$$

which by virtue of (43) and (44) is equal to $2h_1$, or double the moment for the original polynomial (42). And since the sum of the coefficients in any power of (42) will remain unity, $2h_1$ is the lever arm of the square, or the distance of its centre of forces from its first term. Likewise for the third power, the lever arm can be shown to be $3h_1$, and so on. But to make the demonstration general, let us suppose that for the k power of (42) we have found that the lever arm is kh_1. Then in the $(k+1)$th power the total moment about the place of its first coefficient will be

$$a_0 kh_1+a_1(kh_1+\Delta x)+\ldots+a_n(kh_1+n\Delta x),$$

which by (43) and (44) reduces to $(k+1)h_1$, and this is the expression for the lever arm. But we have shown that the lever arm is kh_1 when $k=2$, therefore it is also kh_1 when $k=3$, $k=4$, and so on; so that always, the lever arm of the coefficients in the k power of a polynomial will be k times the lever arm of the coefficients of the polynomial. But the whole interval between the first and last term of the k power is k times the interval between the first and last term of the first power. Hence, as k increases, the lever arm of the product increases in the same ratio as the whole length of the product does, and the extremity of the lever arm, or the centre of parallel forces, divides the length of the product in a constant ratio. This is exemplified in the simple case of the expansion of $(p+q)^m$, or of $(p+qz)^m$, where p and q are probabilities whose sum is unity. The centre of parallel forces in the first power of this binomial, according to (44), is at the distance $h_1=q\Delta x$ measured from the place of the first coefficient p, the whole length of the binomial being Δx. In the expansion to the mth power, the whole length of the expansion is $m\Delta x$, and by the foregoing principle, the distance from the first term of the expansion to its centre of parallel forces is $mh_1 = qm\Delta x$. But we already know that this distance is that of the maximum coefficient in the expansion whenever qm is a whole number (ANALYST, May 1879, p. 66), and that this maximum forms the vertex of the probability curve which is the limit of the series of coefficients when m becomes infinite. If qm is not a whole number, it of course cannot denote the exact rank of any term in the series of coefficients, and the rank of the greatest term will differ from it by a fraction. Formula (3) of the art. just cited is

$$\frac{\Delta y}{y}=\frac{(qm-p)\Delta x-x}{p(x+\Delta x)},$$

where y denotes any term, and x is its distance from the first term. If now

we regard y as a continuously varying function of x or ordinate to a curve, and put $\Delta y = 0$, we get

$$x = (qm-p)\Delta x,$$

showing that the two ordinates whose ranks are

$$qm - p, \qquad \text{and} \qquad qm - p + 1,$$

are always equal to each other. The greatest term must obviously fall between these two. The maximum ordinate to the curve, however, does not in general coincide with the greatest term of the series, nor does it have the precise rank qm. To show this, let us take the general term whose rank reckoned from the first term is i,

$$y_i = \frac{1.2.3\ldots m}{1.2.3\ldots i.1.2.3\ldots(m-i)} p^{m-i} q^i,$$

and apply Stirling's formula

$$1.2.3\ldots m = (2\pi)^{\frac{1}{2}} m^{m+\frac{1}{2}} e^{-m},$$

which gives after reduction

$$y_i = \left(\frac{(2\pi)^{-\frac{1}{2}} m^{m+\frac{1}{2}} p^m}{i^{i+\frac{1}{2}}(m-i)^{m-i+\frac{1}{2}}}\right)\left(\frac{q}{p}\right)^i.$$

Neglecting constant factors and taking the Napierian logarithm, the second member becomes

$$f(i) = i\log'(q \div p) - i\log' i - \tfrac{1}{2}\log' i - (m+\tfrac{1}{2})\log'(m-i) + i\log'(m-i),$$

and the value of i which renders this a maximum will render y_i a maximum. It is given by the condition

$$\frac{df(i)}{di} = \log'\left(\frac{q}{p}\right) + \log'(m-i) - \log' i + \frac{1}{2(m-i)} - \frac{1}{2i} = 0.$$

When we put $i = qm$, this condition reduces to $q-p=0$, indicating that the rank of the maximum ordinate cannot be precisely qm unless $p=q$. If we apply Newton's rule for finding a root of an equation, we get as the approximate rank of the maximum ordinate the value

$$qm + (q-p) \div \left\{ 2 - \frac{1}{m}\left(\frac{q}{p} + \frac{p}{q}\right) \right\},$$

or when m is very large

$$qm + \tfrac{1}{2}(q-p).$$

It is true that Stirling's formula is exact only when m is infinite, but it comes very near the truth when m has any large finite value.

Since the distance from the greatest term in the expansion of $(p+q)^m$ to any other term represents an error or deviation from the most probable value of an observed quantity, and since any term in the expansion is proportional to the number of times which the corresponding error may be expected to occur, our demonstration that the greatest term is at the centre

of parallel forces, whether m be finite or infinite, provided that the rank qm of that centre is an integer, is a proof that in this case the probable sum of all the errors on one side of the most probable value, is equal to the probable sum of all those on the other side. But this is the characteristic of the arithmetical mean, that the sum of the deviations in excess of the mean is equal to the sum of those in defect, or in other words, that the algebraic sum of all the deviations is zero. Thus the property which we have demonstrated gives support to the assumption, which some writers have accepted as an axiom, that in a series of observed values of a single quantity, when the observations are of equal weight, the most probable value of the quantity is the arithmetical mean of all the observed values. (Compare Merriman's *Least Squares*, pp. 127 and 196.)

From what we have seen in connection with (19) and (20), it follows that if the conditions $b_2 = 0$, $b_3 = 0, \ldots . b_n = 0$, or what comes to the same thing, $b_2(\Delta x)^2 = 0$, $b_3(\Delta x)^3 = 0, \ldots . b_n(\Delta x)^n = 0$, are satisfied in the given polynomial when b_2, b_3 &c., are estimated from the centre of parallel forces, the same condtions will be satisfied for the centre of forces in the k power. To illustrate, take the polynomial

$$\tfrac{1}{81}(-4+30z+60z^2-5z^3),$$

whose centre of parallel forces, by (44), is at the distance

$$h_1 = \frac{\Delta x}{81}\Big(30+2\times60-3\times5\Big) = \frac{5}{3}\Delta x,$$

reckoned from the place of the first coefficient, being $\frac{2}{3}$ of the way from the 30 to the 60. For this centre we have not only $b_1\Delta x = 0$, but also in this instance

$$b_2(\Delta x)^2 = \frac{(\Delta x)^2}{81}\left\{-4\Big(-\frac{5}{3}\Big)^2+30\Big(-\frac{2}{3}\Big)^2+60\Big(\frac{1}{3}\Big)^2-5\Big(\frac{4}{3}\Big)^2\right\} = 0,$$

$$b_3(\Delta x)^3 = \frac{(\Delta x)^3}{81}\left\{-4\Big(-\frac{5}{3}\Big)^3+30\Big(-\frac{2}{3}\Big)^3+60\Big(\frac{1}{3}\Big)^3-5\Big(\frac{4}{3}\Big)^3\right\} = 0.$$

Now for example let $k = 3$. The third power of the polynomial is

$$\tfrac{1}{531441}(-64+1440x-7920x^2-16440x^3+122400x^4+317700x^5$$
$$+161700x^6-51750x^7+4500x^8-125x^9),$$

and by the foregoing theorem, its centre of forces should be at the distance $3(\frac{5}{3}\Delta x) = 5\Delta x$ from the first coefficient, and therefore coincident with the 317700. By actual calculation, we find at this centre

$$b_1\,\Delta x = \frac{\Delta x}{531441}\Big\{-64(-5)+1440(-4)-7920(-3)-16440(-2)$$
$$+122400(-1)+161700(1)-51750(2)+4500(3)-125(4)\Big\} = 0,$$

$$b_2(\varDelta x)^2=\frac{(\varDelta x)^2}{531441}\{-64(-5)^2+1440(-4)^2-7920(-3)^2-16440(-2)^2$$
$$+122400(-1)^2+161700(1)^2-51750(2)^2+4500(3)^2-125(4)^2\}=0,$$
$$b_3(\varDelta x)^3=\frac{(\varDelta x)^3}{531441}\{-64(-5)^3+1440(-4)^3-7920(-3)^3-16440(-2)^3$$
$$+122400(-1)^3+161700(1)^3-51750(2)^3+4500(3)^3-125(4)^3\}=0,$$

so that the calculation agrees with the theory.

Let us now pass on to consider another property. In the polynomial (42) suppose that $b_2(\varDelta x)^2$ is not zero, but has some other value. It is required to find its value for the k power of the polynomial. Let us denote the values of b_2 for the 1st, 2nd, 3rd &c., powers by b'_2, b''_2, b'''_2 &c., and let the lever arms of the several systems of coefficients about the first or left hand coefficients as fulcrums be h_1, h_2, h_3 &c. Again, let us suppose that the coefficients in each of these powers represent a system of material points attached to the axis of X and rotating in the plane of XY about the place of the left hand coefficient as an axis, and denote the radii of gyration of the systems for the 1st, 2nd, 3rd &c., powers by g_1, g_2, g_3 &c. This is merely for convenience of description and notation, since material points or masses are positive, while some of the coefficients a may be negative, a circumstance, however, which makes no difference in the analysis. We write then, for the first power,

$$g_1^2=(1^2a_1+2^2a_2+\ldots+n^2a_n)(\varDelta x)^2. \tag{45}$$

But from the nature of b_2 we have

$$\begin{aligned} b'_2(\varDelta x)^2 &= a_0(-h_1)^2+a_1(-h_1+\varDelta x)^2+a_2(-h_1+2\varDelta x)^2+\ldots+a_n(-h_1+n\varDelta x)^2 \\ &= h_1^2(a_0+a_1+\ldots+a_n)-2h_1\varDelta x(a_1+2a_2+\ldots+na_n) \\ &\qquad +(\varDelta x)^2(1^2a_1+2^2a_2+\ldots+n^2a_n), \end{aligned}$$

whence, by virtue of (43) (44) and (45),

$$b'_2(\varDelta x)^2=g_1^2-h_1^2, \tag{46}$$

so that for any polynomial, the value of $b_2(\varDelta x)^2$ is the excess of the square of the radius of gyration above the square of the lever arm, both being reckoned from the left hand term as an axis or fulcrum. In the second power of (42) therefore, we shall have likewise

$$b''_2(\varDelta x)^2=g_2^2-h_2^2. \tag{47}$$

But h_2 is equal to $2h_1$, as already proved. To find g_2, we remember that the sum of the coefficients in the square of (42) is unity, so that the moment of inertia of the system about its first term as an axis is g_2^2, and this moment is equal to the sum of the moments for the portions of the product formed by multiplying (42) by a_0, a_1z, a_2z^2 &c., in succession. We have then

$$g_2^2 = a_0(1^2a_1+2^2a_2+\dots+n^2a_n)(\Delta x)^2+a_1[1^2a_0+2^2a_1+\dots+(n+1)^2a_n] \times(\Delta x)^2+a_2[2^2a_0+3^2a_1+\dots+(n+2)^2a_n](\Delta x)^2+\dots. +a_n[n^2a_0+(n+1)^2a_1+(n+2)^2a_2+\dots+(n+n)^2a_n](\Delta x)^2.$$

In the last or general term, the coefficient of $a_n(\Delta x)^2$, by expansion of the binomials in it, and by virtue of (43) (44) and (45) reduces to

$$n^2+2n\left(\frac{h_1}{\Delta x}\right)+\left(\frac{g_1}{\Delta x}\right)^2,$$

and assigning to n the values 0, 1, 2 &c., in succession, we get expressions for the coefficients of $a_0(\Delta x)^2$, $a_1(\Delta x)^2$ &c., and thus find

$$g_2^2 = a_0g_1^2+a_1[1^2(\Delta x)^2+2.1h_1\Delta x+g_1^2]+a_2[2^2(\Delta x)^2+2.2h_1\Delta x+g_1^2] +\dots.+a_n[n^2(\Delta x)^2+2.nh_1\Delta x+g_1^2],$$

which by means of (43), (44) and (45), reduces to

$$g_2^2 = 2(g_1^2+h_1^2). \tag{48}$$

Substituting this in (47) we get

$$b''_2(\Delta x)^2 = 2(g_1^2-h_1^2),$$

and comparing it with (46), we see that the value of $b_2(\Delta x)^2$ for the second power of the given polynomial is twice as great as it is for the first power. We might proceed to show that for the third power it is three times as great as for the first, and so on. But to make the demonstration general for all powers, let us suppose that for the k power of (42) we have found, as in (48),

$$g_k^2 = k[g_1^2+(k-1)h_1^2]. \tag{49}$$

Let us write also, as in (10),

$$(a_0+a_1z+\dots+a_nz^n)^k = B_0+B_1z+B_2z^2+\dots+B_{kn}z^{kn}.$$

We have obviously, as in (43), (44) and (45),

$$\left.\begin{aligned} &B_0+B_1+B_2+\dots.+B_{kn} = 1,\\ &h_k = (B_1+2B_2+\dots.+knB_{kn})\Delta x,\\ &g_k^2 = [1^2B_1+2^2B_2+\dots.+(kn)^2B_{kn}](\Delta x)^2. \end{aligned}\right\} \tag{50}$$

In the $k+1$ power of (42) we have, for the moment of inertia about the place of the first term as an axis,

$$g_{k+1}^2 = a_0[1^2B_1+2^2B_2+\dots+(kn)^2B_{kn}](\Delta x)^2+a_1[1^2B_0+2^2B_1+\dots. +(kn+1)^2B_{kn}](\Delta x)^2+a_2[2^2B_0+3^2B_1+\dots+(kn+2)^2B_{kn}](\Delta x)^2+\dots +a_n[n^2B_0+(n+1)^2B_1+\dots+(kn+n)^2B_{kn}](\Delta x)^2.$$

In the last or general term, the series is reducible, as before, by expansion of its binomials, and by virtue of (50), to

$$n^2+2n\left(\frac{h_k}{\Delta x}\right)+\left(\frac{g_k}{\Delta x}\right)^2.$$

Assigning to n the values 0, 1, 2 &c., in succession, we get equivalents for the series in the first, second and other terms, whence

$$g_{k+1}^2 = a_0 g_k^2 + a_1[1^2(\Delta x)^2 + 2.1 h_k \Delta x + g_k^2] + a_2[2^2(\Delta x)^2 + 2.2 h_k \Delta x + g_k^2] + \dots + a_n[n^2(\Delta x)^2 + 2.n h_k \Delta x + g_k^2].$$

By means of (43), (44) and (45), this is reduced to

$$g_{k+1}^2 = g_k^2 + g_1^2 + 2h_1 h_k.$$

But we know that $h_k = kh_1$, and assigning to g_k^2 the value in (49), we have

$$g_{k+1}^2 = kg_1^2 + k(k-1)h_1^2 + g_1^2 + 2kh_1^2,$$

and finally

$$g_{k+1}^2 = (k+1)(g_1^2 + kh_1^2).$$

But this expression may be obtained from (49) by merely changing k into $k+1$, whence it appears that if (49) holds good for any one power of the given polynomial, it also holds good for the next higher power. And in (48) it was proved to be true for $k = 2$, therefore it is true for $k = 3$, for $k = 4$, and so on without limit.

Now as was shown in (46), so here, we shall have in the expanded k power

$$b_2^{(k)}(\Delta x)^2 = g_k^2 - h_k^2,$$

where $h_k = kh_1$ and g_k^2 has its value from (49), whence

$$b_2^{(k)}(\Delta x)^2 = kg_1^2 + k(k-1)h_1^2 - k^2 h_1^2,$$

and finally

$$b_2^{(k)}(\Delta x)^2 = k(g_1^2 - h_1^2).$$

Comparing this with (46) we see that the value of $b_2(\Delta x)^2$ for the k power of the given polynomial is k times as great as it is for the first power. It may be noticed that, according to the mechanical analogy which we have employed, the quantity $b_2(\Delta x)^2$ corresponds to the square of the radius of gyration, when the centre of parallel forces is taken as an axis.

As an illustration of the foregoing theorem, take the polynomial

$$\tfrac{1}{4}(2 + 3z - z^2),$$

where by (44) the centre of parallel forces is at the distance

$$h_1 = \tfrac{1}{4}\Delta x(3-2) = \tfrac{1}{4}\Delta x,$$

measured from the first coefficient, and therefore one fourth of the way from the 2 to the 3. We have here

$$b'_2(\triangle x)^2 = \frac{(\triangle x)^2}{4}\left\{ 2\left(-\frac{1}{4}\right)^2 + 3\left(\frac{3}{4}\right)^2 - 1\left(\frac{7}{4}\right)^2 \right\} = -\frac{5}{16}\left(\triangle x\right)^2.$$

Now take for instance $k = 4$, and the fourth power of the polynomial is

$$\tfrac{1}{256}(16 + 96z + 184z^2 + 72z^3 - 111z^4 - 36z^5 + 46z^6 - 12z^7 + z^8).$$

According to the theorem we have established, the centre of paeallel forces here is at the distance $4(\frac{1}{4}\triangle x) = \triangle x$ from the first coefficient, that is,

it coincides with the 96. Also we ought to have

$$b_2^{\mathrm{iv}}(\triangle x)^2 = 4[-\tfrac{5}{16}(\triangle x)^2] = -\tfrac{5}{4}(\triangle x)^2.$$

And by actual computation, we find

$$b_2^{\mathrm{iv}}(\triangle x)^2 = \tfrac{1}{256}(\triangle x)^2[16(-1)^2+184(1)^2+72(2)^2-111(3)^2-36(4)^2$$
$$+46(5)^2-12(6)^2+1(7)^2] = -\tfrac{5}{4}(\triangle x)^2,$$

so that the result confirms the previous work.

As a further illustration, take the binomial $p+q$ or $p+qz$, where p and q are probabilities whose sum is unity. As already shown, the centre of parallel forces is at the distance $q\triangle x$ measured from the first coefficient p. We have then

$$b'_2(\triangle x)^2 = (\triangle x)^2[pq^2+qp^2] = pq(\triangle x)^2.$$

Now in the expansion of $(p+q)^m$ our theorem gives

$$b_2^{(m)}(\triangle x)^2 = pqm(\triangle x)^2.$$

Since $b_2^{(m)}(\triangle x)^2$ is estimated from the centre of parallel forces, which coincides, as we have before seen, with the greatest coefficient or maximum probability, and since it is the sum of the products formed by multiplying each term of the expansion into the square of its distance from the greatest term, we see that it is here identical with ε^2, or the square of the quadratic (mean) error. Hence

$$\varepsilon^2 = pqm(\triangle x)^2, \tag{51}$$

and we have a complete demonstration of the property which I had originally found only by induction from a number of particular cases. (ANALYST, May 1879, p. 71.) As was there shown, when $\triangle x$ is rebuced to dx, and m is increased to an infinity of the second order, ε still remains finite and has the well known value

$$\varepsilon = dx\sqrt{(pqm)}.$$

In conclusion, it may be observed that the lever arm of a system of parallel forces, and the radius of gyration of a system of material points, will remain unchanged when all the forces, or the infinitesimal masses of all the material points, are increased or diminished in a constant ratio. Any rational and entire polynomial is equal to a polynomial the sum of whose coefficients is unity, multiplied by a numerical factor. Hence we shall always have

$$h_k = kh_1, \qquad b_2^{(k)}(\triangle x)^2 = kb'_2(\triangle x)^2,$$

whether the condition (43) is fulfilled or not.

NOTE.—In that portion of the present paper which appeared in the November number, it ought perhaps to have been said respecting the curves given by equations (37) and (41), that the method of integrating between the finite limits $\pm x_1$, instead of $\pm\infty$, is not a very satisfactory one, and

should be taken only as provisional, enabling us to go on and examine the general properties of the curves. The ordinates beyond the limits x_1 are not zero, though they may be small enough to be neglected without material error. I think that the decimal coefficients in (41) are very nearly if not entirely correct, but am not so sure of those in (37), because so far as we can judge from Table III, the values of Y beyond the adopted lower limit $-x_1 = -37$ nearly, will not be very small. It is to be hoped that the integrals of (33) and (40) may yet be evaluated rigorously between the limits $\pm\infty$, so as to give the constants of integration for these curves in an exact form, as has been done in the simpler and well known case of the probability curve (31).

The last one of the three conditions in (40) is satisfied necessarily, because the curve is symmetrical with respect to the axis of Y, and thus its equation (41) shows only two constants of integration. In all cases where the order n of the general differential equation (28) is odd, the curve is symmetrical, the alternate conditions

$$\int_{-\infty}^{\infty} x^3 y dx = 0, \qquad \int_{-\infty}^{\infty} x^5 y dx = 0, \qquad \int_{-\infty}^{\infty} x^7 y dx = 0, \text{ \&c.,}$$

are necessarily satisfied, and the number of constants of integration to be determined is only $\frac{1}{2}(n+1)$.

ON THE VARIATION IN THE LENGTH OF THE DAY.

BY PROFESSOR DANIEL KIRKWOOD.

IF we let m = the mass of a rotating globe in the process of condensation;
t = its time of rotation;
k = its principal radius of gyration;
r = its radius, and
ω = its angular velocity; then the principle of the preservation of areas gives us

$$m\omega k^2 = \tfrac{2}{5} m\omega r^2 = c = \text{a constant.}$$

$$\therefore\ \omega = \frac{5c}{2m} \cdot \frac{1}{r^2}; \text{ or, since } m \text{ is constant,}$$

$$\omega = \frac{c'}{r^2};$$ that is, the angular velocity varies inversely as the square of the radius.